ON HOMOLOGIES OF FINITE PROJECTIVE PLANES†

BY

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ABSTRACT

Study of finite projective planes which have a polarity preserved by many homologies.

1. Introduction

A standard method of studying projective planes is to use perspectives. After presenting a useful but elementary criterion for the transitivity of a finite permutation group, we consider projective planes having many perspectivities. Wagner [16], Piper [14], [15], and Cofman [4] have shown that a finite projective plane is desarguesian if either each point in the plane is the center of a nontrivial elation, or each is the center of a nontrivial homology, but no point or line is fixed by all perspectivities. Dembowski [5, p. 193, footnote] has asked about the possibility of generalizing these results. Using the analogue for projective planes of the Chevalley commutator relations, we obtain information concerning (not necessarily finite) projective planes in which each point is the center of a nontrivial perspectivity but in which no point or line is fixed by all collineations. Specializing to the finite case, we obtain a result which was discovered independently by C. Hering and A. Hoffer: such a plane is desarguesian, except possibly if there is a polarity of the plane preserved by all collineations. By a result of Baer [2], in the exceptional case the plane must have square order.

We then consider the more general situation of a finite projective plane $\mathscr{P}$ and a polarity θ of $\mathscr{P}$ such that each nonabsolute line containing an absolute point is the axis of a nontrivial perspectivity preserving θ. In particular, elations

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are not assumed to be present. It is shown that such a plane is desarguesian if θ is preserved by an involutory perspectivity. From this it follows easily that $\mathscr{P}$ is desarguesian if its order is not a fourth power. Note that it is straightforward to use [10] to determine the structure of collineation groups of planes of even order which are generated by involutory elations and which fix no point. We will therefore deal primarily with planes of odd order.

The major obstacle to the study of collineation groups of finite projective planes are probably Baer involutions. Even if involutory perspectivities are present, only a few results are known which allow one to deal with these unpleasant creatures. A primary goal in the writing of this paper was to try to handle collineation groups which may contain Baer involutions, particularly in the case of planes of odd order. This led to the results presented in Section 2.

We use standard notation, essentially that of [5]. The only unusual notation is $\mathscr{P}_\Sigma$, used to designate the set of fixed points and lines of a set Σ of collineations of a projective plane $\mathscr{P}$.

I am indebted to Alan Hoffer for suggesting problems of the type studied in Section 5.

2. A transitivity criterion

THEOREM 2.1. *Let $\mathscr{P}$ be a finite projective plane of odd order, and Γ a collineation group fixing a line A and a point $a \in A$. Suppose that for some $b \in A - a$. $\Gamma(b)$ contains an involutory homology but no Klein group. Then Γ is transitive on those points $c \in A - a$ which are centers of nontrivial homologies.*

See (3.1) for the reason the assumption was made concerning Klein groups. (2.1) is an immediate consequence of the following simple result on permutation groups, which is a useful companion to Gleason's lemma.

LEMMA 2.2. *Let G be a permutation group on a finite set Ω. Suppose that each point is fixed by a group of prime power order fixing only that point. Suppose further that there is a point $\alpha \in \Omega$ and an involution $t \in G_\alpha$ fixing only α such that t commutes with no G_α-conjugate $\neq t$ of t. Then G is transitive on Ω.*

PROOF. If t and u are conjugate in G and $tu = ut$, then u fixes α, and hence $t = u$ by hypothesis. Hence, by Glauberman's Z^*-theorem [8], $tO(G) \in Z(G/O(G))$. Since $O(G)$ has an odd number of Sylow p-subgroups for any prime p, it follows that t normalizes a Sylow p-subgroup of $O(G)$, and hence also one of G (as $G = O(G)C_G(t)$).

Suppose $\beta \in \Omega - \alpha^G$. By hypothesis, there is a prime p and a p-subgroup of G_β fixing only β. Then $|\beta^G| \equiv 1 \pmod p$, so that a Sylow p-subgroup P of G fixes a unique point of β^G. We may assume that t normalizes P, and hence fixes some point of β^G. Since t fixes only $\alpha \notin \beta^G$, the supposition is absurd.

We will need another application of the preceding lemma.

THEOREM 2.3. *Let Γ be a collineation group of a finite projective plane of odd order. Let L and X be lines, and x and y points of L. Assume that*

(i) *$\Gamma(x, X)$ contains an involutory homology, but no Klein group;*

(ii) *$\Gamma(y)$ contains a nontrivial elation with axis $\neq L$;*

(iii) *$|\Gamma(y)|$ is odd; and*

(iv) *$X \cap L \notin y^\Gamma$.*

Then $\Gamma(L)$ contains an involutory homology, and $x^\gamma = X \cap L$ for some $\gamma \in \Gamma$

PROOF. Suppose $X \cap L \in x^\Gamma$. Then an involution $\sigma \in \Gamma(x, X)$ centralizes some involution $\tau \in \Gamma(X \cap L)$. It is well known that this implies that $\sigma\tau \in \Gamma(L)$ (see (3.1i)).

Next, suppose $X \cap L \notin x^\Gamma$. Then (2.2) applies to Γ acting on $x^\Gamma \cup y^\Gamma$, and this contradicts (iii).

We mention without proof a presumably less useful relative of (2.2).

LEMMA 2.4. *Let G be a permutation group on a finite set Ω. Suppose that, for each $\alpha \in \Omega$, there is a normal subgroup $N(\alpha) \neq 1$ semiregular on $\Omega - \alpha$. If. $|N(\alpha)|$ is even for some α, then G has at most two orbits on Ω.*

We note that there are examples where two orbits actually occur. We leave it to the reader to concoct further applications of (2.2) or (2.4) to the study of elations and homologies of finite planes.

3. Known results

Let $\mathscr{P}$ be a projective plane.

LEMMA 3.1. (*Ostrom and Lüneburg*; see [13, (2.1)].) *Let σ be an involutory (x, X)-homology and τ an involutory (y, Y)-homology. If $X \neq Y$ and $\sigma\tau = \tau\sigma$, then*

(i) *$\sigma\tau$ is an involutory $(X \cap Y, xy)$-homology, and*

(ii) *σ is the only (x, X)-homology.*

COROLLARY 3.2. ([13, (2.2)].) *There is no abelian collineation group of* $\mathscr{P}$ *of order* 8 *generated by three involutory homologies not all having the same axis.*

LEMMA 3.3. *If a subplane of* $\mathscr{P}$ *is fixed by a nontrivial perspectivity* γ, *then the center and axis of* γ *are in the subplane.*

LEMMA 3.4. (*Baer* [2].) *A polarity* θ *of a finite projective plane of order* n *has at least* $n+1$ *absolute points. If this number is* $n+1$, θ *is orthogonal*; *if it is greater than* $n+1$, *then* n *is a square.*

A polarity θ of a finite projective plane is called *regular* if every nonabsolute line containing at least one absolute point contains the same number $s+1$ of absolute points. Call a nonabsolute line *good* if it contains absolute points and *bad* otherwise; a point x is good (or bad) if x^θ is good (or bad). The main facts concerning regular polarities are summarized in (3.5).

LEMMA 3.5. *Let* θ *be a regular polarity of a finite plane of order* n.

(i) ([2]; [5, *p.* 154].) *Each good line contains exactly* $s(n-1)/(s+1)$ *good points and* $(n-s^2)/(s+1)$ *bad points. Each bad line contains exactly* $(sn+1)/(s+1)$ *good points and* $(n+s)/(s+1)$ *bad points. There are exactly* $sn+1$ *absolute points. Moreover,* $s \equiv n (\mathrm{mod}\ 2)$.

(ii) ([7], [11].) *If* α *is a Baer involution preserving* θ, *and* $\mathscr{P}_\alpha$ *is its fixed point subplane, then* θ *induces an orthogonal polarity on* $\mathscr{P}_\alpha$. *Thus,* α *fixes exactly* $\sqrt{n}+1$ *absolute points.*

LEMMA 3.6. *If a collineation* γ *preserves a polarity* θ *and fixes three non-collinear absolute points of* θ, *then* γ *is planar.*

PROOF. If γ fixes the noncollinear absolute points x, y, z, then it fixes the quadrangle $x, y, z, x^\theta \cap y^\theta$.

LEMMA 3.7. *Let* θ *be an orthogonal polarity of a projective plane* $\mathscr{P}$ *of odd order. Suppose that each nonabsolute line containing absolute points is the axis of a nontrivial homology preserving* θ. *Then* $\mathscr{P}$ *is desarguesian.*

PROOF. Let Γ be the group generated by the given involutory homologies. If x is an absolute point, then Gleason's lemma implies that Γ_x is transitive on the remaining absolute points. The result is now an immediate consequence of [13, (2.4)].

LEMMA 3.8. *Let θ be an orthogonal polarity of a projective plane of order n. Let γ be a nontrivial (a, A)-perspectivity preserving θ. Then $A = a^{\theta}$, and one of the following holds.*

(i) *n is odd, $|\gamma| = 2$, and a is nonabsolute.*

(ii) *n is even, and a is absolute.*

(iii) *n is even, and A contains all absolute points.*

The proof of (3.8) is straightforward, and will be omitted.

4. The commutator relations

Given a projective plane and a collineation group Γ, the following result will be denoted $[uUvV]$, where u, v are points and U, V are lines.

$[uUvV]$: If $u \neq v \in U \neq V$, $u \notin V$, $\Gamma(uU) \neq 1$, and $\Gamma(vV) \neq 1$, then $\Gamma(v, U) \neq 1$.

In fact $1 \neq [\Gamma(u, U), \Gamma(v, V)] \leqq \Gamma(v, U)$ (see [5, p. 121]). This statement is the analogue for arbitrary projective planes of the Chevalley commutator relations. In this section we present some straightforward applications of $[uUvV]$ to projective planes which are not necessarily finite.

THEOREM 4.1. *Let Γ be a collineation group of a projective plane $\mathscr{P}$, fixing no point or line, such that each point is the center of a nontrivial perspectivity in Γ. Then one of the following holds.*

(i) *If L is a line for which $\Gamma(L) \neq 1$, then $\Gamma(x, L) \neq 1$ for all $x \in L$.*

(ii) *There is a 1-1 mapping θ from the set of points into the set of lines such that $y \in x^{\theta}$ implies $x \in y^{\theta}$, $\gamma\theta = \theta\gamma$ for all $\gamma \in \Gamma$, and $\Gamma(x, X) \neq 1$ if and only if $X = x^{\theta}$.*

PROOF. Suppose (i) does not hold. Most of the proof will be devoted to obtaining a contradiction from the existence of a line A and distinct points a, a', for which $\Gamma(a, A) \neq 1$ and $\Gamma(a', A) \neq 1$.

First suppose $a, a' \in A$. If $b \in A$, then $\Gamma(b, A) \neq 1$, since if $\Gamma(b, B) \neq 1$ with $B \neq A$, we may assume that $a \notin B$, and then $[aAbB]$ implies that $\Gamma(b, A) \neq 1$. Now let $X \notin A^{\Gamma}$ be any line with $\Gamma(X) \neq 1$, and let $\Gamma(x, X) \neq 1$. Then, as each point of A can be moved off A by some element of Γ, we may assume that $x \notin A$. Let $y = A \cap X$, so $y \neq x$. Then $[xXyA]$ implies that $\Gamma(y, X) \neq 1$. Let $\gamma \in \Gamma$ with $y \notin A^{\gamma}$, and set $y' = A^{\gamma} \cap X \neq y$. Then $\Gamma(y', X) \neq 1$ by $[yXy'A]$. It follows that $\Gamma(z, X) \neq 1$ for all $z \in X$. This is a contradiction, as we have assumed that (i) does not hold.

In particular, if $\Gamma(u, U) \neq 1$ with $u \in U$, then Γ_U fixes u, so $v \in U - u$ and $\Gamma(v, V) \neq 1$ imply that $u \in V$.

Next, consider the possibility $a \in A$, $a' \notin A$, $\Gamma(a, A) \neq 1$, $\Gamma(a', A) \neq 1$. Let $b \in A - a$ and $\Gamma(b, B) \neq 1$, so $a \in B$. Also, $a' \in B$ (otherwise $[a'AbB]$ implies that $\Gamma(b, A) \neq 1$). Thus, $\Gamma(b) = \Gamma(b, aa')$ for all $b \in A - a$. Let $a^\gamma \neq a$, $\gamma \in \Gamma$. Then $\Gamma(a^\gamma, A^\gamma) \neq 1$, so $a^\gamma \notin A$. Similarly, $a \notin A^\gamma$. Using $b = A \cap A^\gamma$, we find that $\Gamma(b, aa') \neq 1$ and $\Gamma(b, a^\gamma a'^\gamma) \neq 1$. Thus, $aa' = a^\gamma a'^\gamma$, so $a^\Gamma \subseteq aa'$, which is not the case.

Finally, suppose that $a \notin A$, $a' \notin A$, $a \neq a'$, $\Gamma(a, A) \neq 1$, and $\Gamma(a', A) \neq 1$. Set $e = A \cap aa'$. If $b \in A - e$ and $\Gamma(b, B) \neq 1$, then $B = aa'$ (if, for example, $a \notin B$, then $\Gamma(b, A) \neq 1$ by $[aAbB]$). Let $e_\gamma \notin A, \gamma \in \Gamma$. Clearly, $\Gamma(a, A)$ or $\Gamma(a', A)$ moves e^γ. Thus, we can find x and X for which $\Gamma(x, X)$ moves e. In particular, $x \neq e \notin X$. By what has been proved for all $b \in A - e$, we know that $x \notin A$ and that the roles of A and $B = aa'$ can be reversed, so $x \notin B$ also. Set $\bar{b} = A \cap X$ and $\bar{a} = B \cap X$, so $\Gamma(\bar{a}, A) \neq 1$ and $\Gamma(\bar{b}, B) \neq 1$. Now $[xX\bar{a}A]$ and $[xX\bar{b}B]$ imply that $\Gamma(\bar{a}, X) \neq 1$ and $\Gamma(\bar{b}, X) \neq 1$, which is not the case.

We have now shown that, for each line L, either $\Gamma(L) = 1$ or $\Gamma(L) = \Gamma(c, L)$ for a unique point c. It remains only to consider the dual situation.

Suppose that $\Gamma(a, A) \neq 1$ and $\Gamma(a, A') \neq 1$, where $a \in A \neq A'$. Let $b \in A$, $b \neq a$, $A \cap A'$, and $\Gamma(b, B) \neq 1$. Then $a \in B$. Consequently, $\Gamma(a, A')$ fixes B and moves b, so $\Gamma(B) \neq \Gamma(b, B)$, which is not the case.

Finally, suppose $\Gamma(a, A) \neq 1$, $\Gamma(a, A') \neq 1$, $a \notin A$, $a \notin A'$, and $A \neq A'$. Let $A \cap A' \neq b \in A$ and $\Gamma(b, B) \neq 1$. Then $a \in B$, so $\Gamma(a, B) \neq 1$ by $[bBaA']$. This contradiction completes the proof of the theorem.

Corollary 4.2. *Let Γ be a collineation group of a projective plane $\mathcal{P}$ such that $\Gamma(x) \neq 1$ and $\Gamma(X) \neq 1$ for all points x and lines X. Then one of the following holds.*

(i) $\Gamma(xX) \neq 1$ *for all X and all $x \in X$.*

(ii) Γ *preserves a polarity θ of $\mathcal{P}$ such that* $\Gamma(x, X) \neq 1$ *if and only if* $X = x^\theta$.

Corollary 4.3. (*Gleason* [9], *Wagner* [16], *Piper* [14].) *A finite projective plane is desarguesian if each point is the center of a nontrivial elation, while no point or line is fixed by all elations.*

Corollary 4.4. (*Wagner* [16], *Piper* [15], *Cofman* [4].) *A finite projective plane is desarguesian if each point is the center of a nontrivial homology, while no point or line is fixed by all homologies.*

COROLLARY 4.5. (*C. Hering, A. Hoffer.*) *Let* Γ *be a collineation group of a finite projective plane* $\mathcal{P}$ *such that* $\Gamma(x) \neq 1$ *for all points* x. *Suppose that* Γ *fixes no point or line. Then* $\mathcal{P}$ *is desarguesian, except possibly if there is a polarity* θ *preserved by* Γ *such that* $\Gamma(x, X) \neq 1$ *if and only if* $X = x^{\theta}$.

PROOF OF (4.3), (4.4.) AND (4.5). Γ is given in (4.5); in (4.3) and (4.4), let Γ be the group generated by the given perspectivities. Then (4.1) applies to Γ. If (4.1ii) holds then θ is a polarity. This is impossible in (4.3) and (4.4), as we would have all points absolute or all nonabsolute.

Suppose (4.1i) holds. Let $\Gamma(X) \neq 1$, $\Gamma(Y) \neq 1$, $x \in X$, $y \in Y$, $x \notin Y$ and $y \notin X$. Set $z = X \cap Y$. By (4.1i), $\Gamma(z, X) \neq 1$ and $\Gamma(z, Y) \neq 1$; hence, both are p-groups for some prime p [9, (1.2)]. For the same reason, $\Gamma(x, X)$ and $\Gamma(y, Y)$ are p-groups. Since both fix xy, Gleason's lemma implies that $\langle \Gamma(x, X), \Gamma(y, Y) \rangle$ can move x to y.

Consequently, since Γ fixes no point or line, it is transitive on the centers of nontrivial elations in Γ. We know that z is the center of nontrivial elations with different axes. Thus, for all $x' \in X$, $\Gamma(x)$ contains elations of order p with different axes. Now [9, (1.6)] readily implies that $\mathcal{P}$ is desargueasian.

REMARK. The above proof of (4.3) and (4.4) is short and elementary; only the preceding argument and the second paragraph of the proof of (4.1) are needed.

It is natural to ask what happens if, in (4.5), we allow a point or line to be fixed by all the given perspectivities. In this case, it is not hard to show that the plane is a translation plane or a dual translation plane.

5. Polarities

In this section we will study the exceptional situation in (4.5) for the case of finite projective planes. More generally, we will consider the following situation.

Hypothesis (#). θ *is a polarity of a finite projective plane* $\mathcal{P}$, A *its set of absolute points, and* Γ *a collineation group preserving* θ *such that* $\Gamma(L) \neq 1$ *for every nonabsolute line* L *meeting* A. *Moreover,* θ *is not orthogonal, and* $\mathcal{P}$ *has order* $q^2 > 4$.

We conjecture that such a plane $\mathcal{P}$ must be desarguesian.

A comment is needed concerning the last sentence of (#). If θ were orthogonal, then $\mathcal{P}$ would have odd order, and hence be desarguesian by (3.7). Thus, there is no loss in assuming that θ is not orthogonal, and then that $\mathcal{P}$ must have square order (3.4). Since a plane of order 4 is desarguesian, we may assume $q^2 > 4$.

Also note that the polarity θ defined in (4.5) is not orthogonal by (3.8).

A major obstacle to the study of ($\#$) is the fact that Γ is not known to be transitive on A, although we cannot prove $\mathscr{P}$ is desarguesian even when θ is defined as in (4.5) and Γ is transitive on A. As we will see, $\mathscr{P}$ is desarguesian if Γ is transitive on the nonabsolute lines meeting A.

LEMMA 5.1. *Assume* ($\#$). *Then* Γ *acts faithfully on each of its orbits on* A.

PROOF. Let A' be such an orbit, and consider $\Gamma(A')$. As Γ fixes no point or line, $\Gamma(A')$ is planar by (3.6). Since there are q^2 nonabsolute lines through each absolute point, $\Gamma(A') = 1$ by (3.3).

THEOREM 5.2. *Assume* ($\#$). *Then* Γ *has no nontrivial solvable normal subgroup.*

PROOF. Deny this. Then Γ has a nontrivial normal elementary abelian p-subgroup Δ for some prime p.

We first show that Δ fixes no point of $\mathscr{P}$. For suppose Δ fixes y. Since $q^2 > 3$ we find, using the homologies in Γ, that y^Γ contains a quadrangle. As Δ fixes each point of y^Γ, Δ is planar. Moreover, Γ acts on $\mathscr{P}_\Delta$. This contradicts (3.3).

We next show that Δ is semiregular on A. For suppose $\Delta_x \neq 1$ for some $x \in A$. Since Δ centralizes Δ_x while fixing no point of $\mathscr{P}$, Δ_x fixes a triangle of absolute points, and hence is planar by (3.6). There is a nonabsolute line M on x not in $\mathscr{P}_{\Delta x}$. Then $\Gamma(M)$ normalizes Δ_x and hence acts on $\mathscr{P}_{\Delta x}$. Once again, by (3.3), this is impossible.

Let $x \in A$ and $1 \neq \delta \in \Delta$, so $x^\delta \neq x$. We claim that δ fixes $L = xx^\delta$. In fact, if $1 \neq \gamma \in \Gamma(L)$ then $x^\delta = x^{\delta\gamma} = x^{\gamma^{-1}\delta\gamma}$, so $\gamma^{-1}\delta\gamma\delta^{-1} \in \Delta_x = 1$. Consequently, δ commutes with γ and hence fixes L. In particular, Δ_L is transitive on $x^\Delta \cap L$.

Consider the set $\mathscr{P}_\delta$ of fixed points and lines of δ. Each absolute point is on a line of $\mathscr{P}_\delta$. However, $\mathscr{P}_\delta \cap A = \varnothing$, so $\mathscr{P}_\delta$ is not a subplane by (3.4). Moreover, Δ acts on $\mathscr{P}_\delta$ and fixes no point of $\mathscr{P}_\delta$. Thus, $\mathscr{P}_\delta$ can only be a triangle with nonabsolute vertices a, b, c. Since Δ fixes $\{a,b,c\}$, it follows that $p = 3$ and $|\Delta| = 9$.

Each line L on $x \in A$ containing two points of x^Δ contains three such points, and Δ_L is transitive on these points. Thus, there are just four such lines L, one for each subgroup of Δ of order 3. Since $q^2 > 6$, there are distinct nonabsolute lines M and M' on x with $M \cap x^\Delta = M' \cap x^\Delta = \{x\}$. Then $\Gamma(M)$ and $\Gamma(M')$ act on $x^\Delta - \{x\}$ semiregularly, so $|\Gamma(M)|$ and $|\Gamma(M')|$ are even. Let $\sigma \in \Gamma(M)$ and $\sigma' \in \Gamma(M')$ be involutions. Then both involutions invert Δ, so $\sigma\sigma'$ fixes each

$y \in x^{\Delta}$. However, if we choose $y \in x^{\Delta} - \{x\}$, then $M^{\theta}, y, y^{\sigma}$ are collinear, as are $(M')^{\theta}, y, y^{\sigma}$ (since $y^{\sigma} = y^{\sigma'}$), so $y \in M^{\theta}M'^{\theta} = x^{\theta}$. Since $x^{\theta} \cap A = \{x\}$, we have arrived at a contradiction.

COROLLARY 5.3. *Γ has no nontrivial normal subgroup of odd order, and no nontrivial normal 2-subgroup.*

PROOF. This is immediate by (5.2) and [6].

THEOREM 5.4. *Assume (#). Then $\mathcal{P}$ is desarguesian if Γ contains an involutory perspectivity.*

PROOF. Suppose first that q^2 is even, so $|\Gamma(x)|$ is even for some $x \in A$. By (5.1), Γ acts faithfully on $A' = x^{\Gamma}$. By (5.3) and [10], Γ has a normal subgroup $\Delta \approx PSL(2,2^e)$, $Sz(2^e)$, or $PSU(3,2^e)$ for some e, where Δ acts on A' in its usual 2-transitive representation. Since A' is not contained in any line, the points of A' and the lines L with $|L \cap A'| \geqq 2$ form a design with $\lambda = 1$ and some k. Moreover, $2^e | k-1$, so $PSL(2,2^e)$ and $Sz(2^e)$ cannot occur. If there is a nonabsolute line M on x with $M \cap A' = \{x\}$, then $\Gamma(M)$ acts semiregularly on $A' - \{x\}$, whereas $|A' - \{x\}| = 2^{3e}$ and $|\Gamma(M)| \mid q^2 - 1$. Thus, each of the q^2 nonabsolute lines meeting A' contains k points of A'. Using $\Gamma(x)$, we find that if $x \in L$, then $L \cap A'$ is a union of lines of the usual design associated with Δ. Since Δ_x permutes the latter lines 2-transitively, it follows that $2^{2e} = q^2$. Hence, $\mathcal{P}$ is desarguesian by [12].

Now suppose q^2 is odd. There is a line M for which $|\Gamma(M)|$ is even. Clearly, M is not absolute.

We claim that there is no nonabsolute point x for which $\Gamma(x)$ contains a Klein group. For, suppose there is such an x. Let y^{θ} be a nonabsolute line on x meeting A, so $y \in x^{\theta}$. (To verify that there is such a line, note that otherwise we would have $q^2 + 1 \geqq |A|$, so θ would be orthogonal by (3.4).) By hypothesis, $\Gamma(y) \neq 1$. Set $L = xy$, so $y = x^{\theta} \cap L$. Note that $\Gamma(x)$ normalizes $\Gamma(y)$, so by (3.1ii) $|\Gamma(y)|$ is odd. In particular, $y \notin x^{\Gamma}$. Let T be the orbit of x under Γ_L. Then $\Gamma(x)$ is semiregular on $T - \{x\}$, so $|T| - 1$ is a power of 2 by [10]. However, $\Gamma(y) \leqq \Gamma_{xL}$ also acts semiregularly on $T - \{x\}$, whereas $|\Gamma(y)|$ is odd. This contradiction proves our claim.

Suppose $M \cap A = \varnothing$. There is a nonabsolute line L on M^{θ} meeting A. By (3.1), (2.3) applies to Γ_L, using $x = M^{\theta}$ and $y \in L \cap A$. Consequently, $|\Gamma(L)|$ is even.

Thus, we assume that there is a point $a \in M \cap A$. By (3.1), (2.1) applies to $\Gamma_{(a)}$

acting on a^θ. Thus, Γ_a is transitive on the q^2 lines other than a^θ on a whenever one of these lines is the axis of an involutory homology. Since each absolute point is on such a line through a, it follows that Γ is transitive on the nonabsolute lines meeting A. Consequently, each such line contains the same number of absolute points, so that θ is a regular polarity.

We will use the notation and terminology of (3.5), where $n = q^2$ and $s > 1$. If $s = q$, (5.4) follows from [13, (6.5)]. We may thus assume that $s < q$, so there are bad lines. Call $\gamma \in \Gamma$ good (or bad) if it is an involutory homology with good (or bad) axis.

If B is a bad line, by (3.5i) it has more good points than bad points. Thus, B contains a good point x for which $x^\theta \cap B$ is also good. Since $|\Gamma(x)|$ and $|\Gamma(x^\theta \cap B)|$ are known to be even, $\Gamma(B)$ contains a bad involution by (3.1i).

Similarly, if L is a good line, it contains a good point x for which $x^\theta \cap L$ is good. Consequently, Γ contains a Klein group whose involutions are all good. We may assume that Γ is generated by its involutory homologies. By (3.1 ii) and the transitivity on good lines, all good involutions are conjugate. Also, each involutory homology is the product of two good involutions. It follows that Γ has no normal subgroup of index 2.

In particular, Γ induces only even permutations on A. Since a bad involution fixes no point of A, $|A| = sq^2 + 1 \equiv 0 \pmod 4$.

We claim that Γ contains no Baer involution α. For, let α be such an involution. By (3.5ii), $(sq^2 + 1) - (q + 1) \equiv 0 \pmod 4$, so $q \equiv 3 \pmod 4$. However the total number of points moved by α is then $(q^4 + q^2 + 1) - (q^2 + q + 1) \equiv 2 \pmod 4$, which is impossible.

Consequently, by (3.2), Γ has no elementary abelian subgroup of order 8. On the other hand, $O(\Gamma) = 1$, Γ has no central involution, and Γ has no normal subgroup of index 2. Consequently, a result of Alperin [1, Prop. 1, its proof, and the remark following its proof] implies that all involutions in Γ are conjugate, which is ridiculous.

COROLLARY 5.5. *Assume* (#). *Suppose that, if α is a Baer involution in Γ, then θ induces an orthogonal polarity on $\mathscr{P}_\alpha$. Then $\mathscr{P}$ is desarguesian.*

PROOF. By (5.3) and a theorem of Brauer and Suzuki [3], Γ contains a Klein group $\langle \alpha, \beta \rangle$. By (5.4), we may assume that α, β and $\alpha\beta$ are Baer involutions. Since β induces either 1 or an involution on $\mathscr{P}_\alpha$, $\langle \alpha, \beta \rangle$ fixes a line L such that, if n is even, then L is nonabsolute and contains just one absolute point of θ lying in $\mathscr{P}_\alpha \cap \mathscr{P}_\beta$. Then $\langle \alpha, \beta \rangle$ acts on the nontrivial group $\Gamma(L)$ of odd order. We may thus assume $C_{\Gamma(L)}(\alpha) \neq 1$. Since $C_{\Gamma(L)}(\alpha)$ acts faithfully on $\mathscr{P}_\alpha$, this contradicts (3.8).

COROLLARY 5.6. *Assume* ($\#$). *Then* $\mathscr{P}$ *is desarguesian if* q^2 *is not a fonrth power.*

PROOF. (3.4) and (5.5).

COROLLARY 5.7. *Assume* ($\#$). *If* Γ *is transitive on those nonabsolute lines which meet* A, *then* $\mathscr{P}$ *is desarguesian.*

PROOF. (3.5 ii) and (5.5).

THEOREM 5.8. *Let* $\mathscr{P}$ *be a finite projective plane each point of which is the center of a nontrivial perspectivity. Suppose that no point or line is fixed by all collineations. Then* $\mathscr{P}$ *is desarguesian if either* (i) *the order of* $\mathscr{P}$ *is not a fourth power, or* (ii) $\mathscr{P}$ *has an involutory perspectivity.*

PROOF. (4.5), (3.7), (5.4), and (5.5).

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